VECTOR CALCULUS

(contd)

B.Sc. Part II

4th Paper

Sums 1. Find the volume of the parallelepiped whose edges are represented by $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

Solution

Volume of a parallelepiped

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) \left[= [\vec{a}\ \vec{b}\ \vec{c}] \right]$$

$$\vec{b}\times\vec{c} = (\vec{i} + 2\vec{j} - \vec{k}) \times (3\vec{i} - \vec{j} + 2\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$\Rightarrow \vec{b}\times\vec{c} = 3\vec{i} - 5\vec{j} - 7\vec{k}$$

$$\therefore \vec{a}\cdot(\vec{b}\times\vec{c}) = (2\vec{i} - 3\vec{j} + 4\vec{k})\cdot(3\vec{i} - 5\vec{j} - 7\vec{k})$$

$$= (2\times3 + 3\times5 - 4\times7)$$ ~~cubic unit~~

$$= -7$$ ~~cubic units~~

$\therefore$ Volume of the parallelepiped $= 7$ cubic unit.

Q. Find the constant k such that

$\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$,

$\vec{c} = 3\vec{i} + a\vec{j} + 5\vec{k}$ are coplanar.

Soln $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow$ scalar triple product $= 0$

$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

~~Now~~ $\Rightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} = 0$ —— (1)

Now, $\vec{a} \times \vec{b} = (2\vec{i} - \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - 3\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$\Rightarrow \vec{a} \times \vec{b} = \vec{i} + 7\vec{j} + 5\vec{k}$

Now $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{i} + 7\vec{j} + 5\vec{k}) \cdot (3\vec{i} + a\vec{j} + 5\vec{k})$

$= 3 + 7a + 5 \times 5 = 28 + 7a$

From (1), $28 + 7a = 0$

$\Rightarrow a = -4$

Q. If $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ then prove that

(i) $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + c\times c' = \vec{0}$

(ii) $\vec{a}'\times\vec{b}' + \vec{b}'\times\vec{c}' + \vec{c}'\times\vec{a}' = \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Soln

(i) LHS $= \vec{a}\times\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} + \vec{b}\times\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$+ \vec{c}\times\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$= \dfrac{\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$= \dfrac{\cancel{(\vec{a}\cdot\vec{c})\vec{b}} - \cancel{(\vec{a}\cdot\vec{b})\vec{c}} + \cancel{(\vec{b}\cdot\vec{a})\vec{c}} - \cancel{(\vec{b}\cdot\vec{c})\vec{a}} + \cancel{(\vec{c}\cdot\vec{b})\vec{a}} - \cancel{(\vec{c}\cdot\vec{a})\vec{b}}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$= \vec{0}$ = RHS

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(ii) $\vec{b}' \times \vec{c}'$

$$= \frac{(\vec{c}\times\vec{a}) \times (\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{[\vec{c}\ \vec{a}\ \vec{b}]\vec{a} - [\vec{a}\ \vec{a}\ \vec{b}]\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]\vec{a} - 0}{[\vec{a}\ \vec{b}\ \vec{c}]^2} = \frac{\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$\therefore\ \vec{b}' \times \vec{c}' = \dfrac{\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$. Similarly, we have

$$\vec{c}' \times \vec{a}' = \frac{\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\vec{a}' \times \vec{b}' = \frac{\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Adding we have

$$\vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' + \vec{a}' \times \vec{b}' = \frac{\vec{a} + \vec{b} + \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Proved